Double-graded supersymmetric quantum mechanics

Andrew James Bruce (iD), and Steven Duplij (iD)

View Online   Export Citation   CrossMark

ARTICLES YOU MAY BE INTERESTED IN

Journal of
Mathematical Physics

READ TODAY!

Special Issue: XIXth International
Congress on Mathematical Physics

AIP Publishing

Double-graded supersymmetric quantum mechanics

Andrew James Bruce[1,a] (iD) and Steven Duplij[2,b] (iD)

AFFILIATIONS

[1] Mathematics Research Unit, University of Luxembourg, Maison du Nombre 6, avenue de la Fonte,
L-4364 Esch-sur-Alzette, Luxembourg
[2] Center for Information Processing ZIV, Universität Münster, D-48149 Münster, Germany

[a] Electronic mail: andrewjamesbruce@googlemail.com
[b] Author to whom correspondence should be addressed: douplii@uni-muenster.de
URL: http://homepages.spa.umn.edu/~duplij

ABSTRACT

A quantum mechanical model that realizes the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded generalization of the one-dimensional supertranslation algebra is proposed. This model shares some features with the well-known Witten model and is related to parasupersymmetric quantum mechanics, though the model is not directly equivalent to either of these. The purpose of this paper is to show that novel "higher gradings" are possible in the context of non-relativistic quantum mechanics.

Published under license by AIP Publishing. https://doi.org/10.1063/1.5118302

I. INTRODUCTION

Supersymmetry in the context of string theory and quantum field theory has a long history dating back to the 1970s[1–3] (for the history and development of supersymmetry, see, e.g., Refs. 4 and 5). However, we know that if Nature does indeed utilize supersymmetry, then supersymmetry must be broken. We have in mind here models of high energy physics. Supersymmetry can also be realized in condensed matter physics (see Ref. 6 and later works). Understanding the possible mechanisms for breaking supersymmetry is vital to construct realistic theories.

In 1981, Witten[7] introduced (non-relativistic) supersymmetric quantum mechanics as a toy model to discuss supersymmetry breaking via instanton effects. This model is the simplest non-trivial model that exhibits the general features of a supersymmetric field theory. Since then the subject has grown, and many useful tools in quantum mechanics have been developed with supersymmetric quantum mechanics as their root.

In particular, Witten's model provides an alternative scheme to the factorization method, gives rise to an understanding of exactly solvable potentials via shape invariance, has applications in the inverse scattering method, etc. (see, e.g., Refs. 8–11). For the most clear exposition of Witten's theory, see Ref. 12, followed by the many papers on "form-invariance."

Since Witten's original work,[7] there have been several generalizations of his quantum mechanical model including parasupersymmetric quantum mechanics,[13] orthosupersymmetric quantum mechanics,[14] and fractional supersymmetric quantum mechanics[15] all of which are motivated by generalizations of standard statistics. Independently, the use of para-Grassmann variables in mechanics has been implemented by Gershun and Tkach.[16,17]

Here, we examine another generalization in which the graded structure of the theory is different from that of the standard supersymmetric quantum mechanics. In particular, we employ a "double-grading" using $\mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2$ rather than just the standard grading via $\mathbb{Z}_2$.

Recently, in Ref. 18, it was proposed a generalization of the $\mathcal{N}$-extended supersymmetry algebra to the setting of $\mathbb{Z}_2^n$-Lie algebras introduced in Refs. 19 and 20. The main aim of Ref. 18 was to establish a geometric understanding of the algebra in terms of $\mathbb{Z}_2^n$-manifolds (see

Refs. 21 and 22), i.e., to build a generalization of super-Minkowski space-time (see also Ref. 23). Loosely, $\mathbb{Z}_2^n$-manifolds are "manifolds" for which the structure sheaf has a $\mathbb{Z}_2^n$-grading, and the commutation rules for the local coordinates come from the standard scalar product. The case of $n = 1$ is just the theory of standard supermanifolds.[24–26] However, in Ref. 18, no examples of classical or quantum systems that have the proposed $\mathbb{Z}_2^n$-Lie algebra as a symmetry were given. We rectify this omission here.

We present a *double-graded* supersymmetric quantum mechanical model on the real line that exhibits the required $\mathbb{Z}_2^n$-symmetry for the case of $n = 2$. This model resembles Witten's version of supersymmetric quantum mechanics[7] (see also Refs. 27 and 28), as well as various models with extended supersymmetry.[12,29–32] However, the differences are as follows:

- the underlying algebra will be a $\mathbb{Z}_2^2$-Lie algebra and not a super Lie algebra (i.e., a $\mathbb{Z}_2$-Lie algebra),
- the Hilbert space will be $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^4$ in order to take account of the four possible "spin" states, and
- we will allow for a central charge.

We remark that $\mathbb{Z}_2^2$-gradings appear in several guises within mathematical physics, for example, in relation to the symmetries of the Lévy-Leblond equation[33] and parastatistics[34,35] (see also Ref. 36). We will show how the models presented here are related to but not equivalent to models that possess parasupersymmetries[13] [see Refs. 16 and 37–39 on a *filtered* generalization of Lie superalgebras (see Refs. 40–43)].

The reader should observe that Scheunert proved a theorem reducing "colored" Lie algebras to either Lie algebras or Lie superalgebras,[44] while Neklyudova proved an analog of this theorem for "colored" graded-commutative and associative algebras.[45] However, these theorems deal only with algebras, not with pairs (algebra, module over it), and for such pairs, the direct analogs of the Neklyudova–Scheunert theorem do not hold. This fact is an indicator that our construction, as well as Refs. 21 and 22 (and the earlier works[46,47]), is not meaningless.

In Sec. II, we present the 1-dimensional $\mathbb{Z}_2^2$-supertranslation algebra and we define quantum mechanical systems that are $\mathbb{Z}_2^2$-supersymmetric and derive some direct consequences of this definition. We will assume that the reader is already somewhat familiar with Witten's model. In Sec. III, we define and examine a specific model that is akin to Witten's model. Some closing remarks are in Sec. IV.

II. PRELIMINARIES

A. $\mathbb{Z}_2^2$-Lie algebras

Let us recall the notion of a $\mathbb{Z}_2^2$-Lie algebra (see Refs. 44 and 48). The extension to $\mathbb{Z}_2^n$-Lie algebras is straightforward, though, in this article, we will not consider the general case. A $\mathbb{Z}_2^2$-graded vector space is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) that is the direct sum of homogeneous vector spaces,

$$\mathfrak{g} = \mathfrak{g}_{00} \oplus \mathfrak{g}_{01} \oplus \mathfrak{g}_{11} \oplus \mathfrak{g}_{10}.$$

Note that we have chosen a particular ordering for the elements of $\mathbb{Z}_2^2 := \mathbb{Z}_2 \times \mathbb{Z}_2$. This ordering will be convenient for our applications in quantum mechanics. We will denote the $\mathbb{Z}_2^2$-degree of an element of $\mathfrak{g}$, a (say), as $\deg(a) \in \mathbb{Z}_2^2$. We define the *even subspace* of $\mathfrak{g}$ to be $\mathfrak{g}_{00} \oplus \mathfrak{g}_{11}$ and the *odd subspace* of $\mathfrak{g}$ to be $\mathfrak{g}_{01} \oplus \mathfrak{g}_{10}$. That is, we pass to a $\mathbb{Z}_2$-grading via the total degree, i.e., the sum of the components of the $\mathbb{Z}_2^2$-degree. We will denote the standard scalar product on $\mathbb{Z}_2^2$ by $\langle -, - \rangle$. That is, if $\deg(a) = (\gamma_1, \gamma_2)$ and $\deg(b) = (\gamma'_1, \gamma'_2)$, then $\langle \deg(a), \deg(b) \rangle = \gamma_1 \gamma'_1 + \gamma_2 \gamma'_2$.

A $\mathbb{Z}_2^2$-*Lie algebra* is a $\mathbb{Z}_2^2$-graded vector space equipped with a bi-linear operation, $[-, -]$, such that for homogeneous elements a, b and $c \in \mathfrak{g}$, the following are satisfied:

1. $\deg([a, b]) = \deg(a) + \deg(b)$,
2. $[a, b] = -(-1)^{\langle \deg(a), \deg(b) \rangle} [b, a]$, and
3. $[a, [b, c]] = [[a, b], c] + (-1)^{\langle \deg(a), \deg(b) \rangle} [b, [a, c]]$.

Extension to inhomogeneous elements is via linearity.

Remark 1. We have written the Jacobi identity for a $\mathbb{Z}_2^2$-Lie algebra in Loday–Leibniz form (see Ref. 49). Note that this form has a direct interpretation independently of the symmetry of the bracket. In particular, the notion of a $\mathbb{Z}_2^2$-Loday–Leibniz algebra is clear, i.e., just drop the symmetry condition.

B. The $\mathbb{Z}_2^2$-supertranslation algebra in one dimension

The starting place is the following $\mathbb{Z}_2^2$-graded Lie algebra (see Refs. 44 and 48) with generators H_{00}, Q_{01}, Q_{10}, and Z_{11} of $\mathbb{Z}_2^2$-degrees $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$, respectively, given by

$$[Q_{01}, Q_{01}] = [Q_{10}, Q_{10}] = \frac{1}{2} H_{00}, \quad [Q_{10}, Q_{01}] = \frac{1}{2} Z_{11}, \tag{1}$$

where all other Lie brackets vanish—this belongs to the $C(2, s)$-family of generalized Lie algebras following Ref. 48. The above algebra was proposed by Bruce[18] and will be referred to as the $\mathbb{Z}_2^2$-*supertranslation algebra (in one dimension)*. Up to conventions, there are essentially

two copies of supersymmetry in one-dimension and a central extension. However, one must take care as we have $\mathbb{Z}_2^2$-graded Lie brackets, i.e., we employ a novel double-grading. This algebra should be compared with the extended SUSY algebra with central charges given in Refs. 50–52.

Let us assume that we can find a representation of this $\mathbb{Z}_2^2$-Lie algebra as an operator on some Hilbert space—the space of states of some quantum system—with respect to the $\mathbb{Z}_2^2$-graded commutator. Explicitly, if we have two homogeneous operators, X and Y, then the commutator is defined as

$$[X, Y] := X \circ Y - (-1)^{\langle \deg(X), \deg(Y) \rangle} Y \circ X. \tag{2}$$

It is easy to verify that the $\mathbb{Z}_2^2$-graded commutator satisfies the requirements to define a $\mathbb{Z}_2^2$-Lie algebra. Extension to inhomogeneous operators is via linearity. Furthermore, we will take the $\mathbb{Z}_2^2$-supercharges Q_{01} and Q_{10} to be Hermitian. Thus, the operator H_{00} is also Hermitian, naturally, this will have the interpretation as a *Hamiltonian*, while the *central charge* Z_{11} will be anti-Hermitian. From these considerations, we are led to the following definition:

Definition 2. A $\mathbb{Z}_2^2$-*supersymmetric quantum mechanical system* ($\mathbb{Z}_2^2$-*SUSY QM system*) *or a double-graded supersymmetric quantum mechanical system is the quintuple,*

$$\{\mathcal{H}; H_{00}, Q_{01}, Q_{10}, Z_{11}\},$$

where

1. $\mathcal{H} = \mathcal{H}_{00} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{11} \oplus \mathcal{H}_{10}$, is a $\mathbb{Z}_2^2$-graded Hilbert space, and
2. H_{00}, Q_{01}, and Q_{10} are Hermitian operators, and Z_{11} is an anti-Hermitian operator of all of the indicated $\mathbb{Z}_2^2$-degrees that satisfy the $\mathbb{Z}_2^2$-supertranslation algebra (1) with respect to the $\mathbb{Z}_2^2$-graded commutator (2).

Remark 3. By defining $Z_{11} = i\overline{Z}_{11}$, where $\overline{Z}_{11}$ is Hermitian, one can replace the anti-Hermitian central charge with a Hermitian one. This will make the physical relevance of the central charge clearer, but it is of little consequence for the mathematical results of this paper.

Directly from the definition of a double-graded supersymmetric quantum mechanical system, we can derive some results that apply for all such systems.

Theorem 4. *For any $\mathbb{Z}_2^2$-SUSY QM system, the expectation value of the Hamiltonian operator H_{00} is positive.*

Proof. This follows in exactly the same way as standard supersymmetric quantum mechanics. In particular, as we have Hermitian charges, we can write $4Q_{\mathbf{i}}^\dagger Q_{\mathbf{i}} = H_{00}$, here, $\mathbf{i} = 01$ or 10. Then, for any state

$$\langle \psi | \frac{1}{4} H_{00} | \psi \rangle = \langle \psi | Q_{\mathbf{i}}^\dagger Q_{\mathbf{i}} | \psi \rangle = \| Q_{\mathbf{i}} | \psi \rangle \|^2 \ \geq 0.$$

$\square$

That is, just as in standard supersymmetric quantum mechanics, the energy is always positive for any $\mathbb{Z}_2^2$-SUSY QM system.

By definition, a $\mathbb{Z}_2^2$-SUSY QM system has good $\mathbb{Z}_2^2$-supersymmetry if and only if there exists at least one state that is annihilated by both $\mathbb{Z}_2^2$-supercharges Q_{01} and Q_{10}. Clearly, this implies that such states have zero energy as $H_{00} \sim Q_{01}^2 = Q_{10}^2$. Otherwise, we say that $\mathbb{Z}_2^2$-supersymmetry is broken. Moreover, it is clear from $Z_{11} \sim [Q_{10}, Q_{01}]$ that zero energy states are also annihilated by the central charge. We have thus proved the following:

Proposition 5.

1. *A $\mathbb{Z}_2^2$-SUSY QM system has good $\mathbb{Z}_2^2$-supersymmetry if and only if the vacuum state has zero energy, otherwise the energy is strictly positive and*
2. *if a $\mathbb{Z}_2^2$-SUSY QM system has good $\mathbb{Z}_2^2$-supersymmetry, then the vacuum state is annihilated by the central charge.*

C. The relation with parasupersymmetry

In the context of parasupersymmetric quantum mechanics (see Refs. 13 and 37), Beckers and Debergh[38] defined, via Green's ansatz (see Ref. 34), the following algebra (here, $\{-, -\}$ is the usual anticommutator and $[-, -]$ the usual commutator):

$$\{Q_i, Q_i\} = \{Q_i^\dagger, Q_i^\dagger\} = 0, \qquad\qquad \{Q_i, Q_i^\dagger\} = H, \tag{3}$$
$$[Q_i, Q_j] = [Q_i, Q_j^\dagger] = [Q_i^\dagger, Q_j^\dagger] = 0 \ (i \neq j), \qquad\qquad [H, Q_i] = 0,$$

for $i, j = 1, 2$.

Proposition 6. There exists a non-canonical morphism from the Beckers–Debergh algebra (3) to the $\mathbb{Z}_2^2$-supertranslation algebra (1) with $Z_{11} = 0$.

Proof. We define $Q_{01} := \frac{1}{2}(Q_1 + Q_1^\dagger)$, $Q_{10} := \frac{1}{2}(Q_2 + Q_2^\dagger)$, and $H_{00} := H$. Note that this is a choice and we could equally have made the other obvious choice here with Q_{01} and Q_{10}. Direct calculation shows that we recover the algebra (1) with $Z = 0$,

$$
\begin{aligned}
[Q_{01}, Q_{01}] &= \frac{1}{2}(Q_1 + Q_1^\dagger)(Q_1 + Q_1^\dagger) \\
&= \frac{1}{2}\{Q_1, Q_1^\dagger\} = \frac{1}{2}H_{00}, \\
[Q_{10}, Q_{10}] &= \frac{1}{2}(Q_2 + Q_2^\dagger)(Q_2 + Q_2^\dagger) \\
&= \frac{1}{2}\{Q_2, Q_2^\dagger\} = \frac{1}{2}H_{00}, \\
[Q_{10}, Q_{01}] &= \frac{1}{4}\big([Q_2, Q_1] + [Q_2, Q_1^\dagger] \\
&\quad +[Q_2^\dagger, Q_1] + [Q_2^\dagger, Q_1^\dagger]\big) = \frac{1}{2}Z_{11} = 0, \\
[Q_{01}, H_{00}] &= \frac{1}{2}\big([Q_1, H] + [Q_1^\dagger, H]\big) = 0, \\
[Q_{10}, H_{00}] &= \frac{1}{2}\big([Q_2, H] + [Q_2^\dagger, H]\big) = 0,
\end{aligned}
$$

where we have directly used the algebra (3) and the fact that H is Hermitian. $\qquad\square$

We stress the point that this morphism is not invertible—there is no way to uniquely decompose a Hermitian operator into the sum of a non-Hermitian operator and its Hermitian conjugate. We also draw attention to the fact that the central charge has to vanish in order to construct the above morphism. Thus, clearly, we do not have an isomorphism between the Beckers–Debergh algebra and the $\mathbb{Z}_2^2$-supertranslation algebra. In other words, although there are clear similarities between the $\mathbb{Z}_2^2$-supertranslation algebra and parastatistics/parasupersymmetry, the two concepts are not the same.

D. "Higher" Pauli matrices

The $\mathbb{Z}_2^2$-degrees of freedom for a given quantum mechanical are given by vectors in $\mathbb{C}^4$ and not just $\mathbb{C}^2$ (i.e., we have more than just spin "up" and "down"). Thus, we cannot directly use the Pauli matrices. Instead, we have to use the so-called "sigma" and "alpha" matrices, which are built by placing the Pauli matrices (and the identity matrix) on the diagonal and anti-diagonal, respectively, i.e., we use the direct sum and the skew sum of matrices. That is, we define $\Sigma_i := \sigma_i \oplus \sigma_i$ and $\alpha_i := \sigma_i \ominus \sigma_i$ ($i = 0, 1, 2, 3$) where σ_i are the Pauli matrices ($\sigma_0 := \mathrm{id}_{2\times 2}$). Explicitly, we have the following matrices:

$$
\Sigma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad
\Sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
\Sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad
\Sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$

and

$$
\alpha_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad
\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

The algebraic properties of these matrices follow from those of the Pauli matrices [see, for example, Ref. 53 (pp. 209–212)]. In particular, if we define $[A, B]_\mp = AB \mp BA$ and follow the definitions, we obtain

$$[\Sigma_i, \Sigma_j]_\mp = [\alpha_i, \alpha_j]_\mp = [\sigma_i, \sigma_j]_\mp \oplus [\sigma_i, \sigma_j]_\mp, \tag{4}$$

$$[\Sigma_i, \alpha_j]_\mp = [\sigma_i, \sigma_j]_\mp \ominus [\sigma_i, \sigma_j]_\mp. \tag{5}$$

Observe that the (anti-)commutators of the Σ and α matrices are completely determined by the (anti-) commutators of the Pauli matrices. We consider the Σ and α matrices to carry the $\mathbb{Z}_2^2$-degree, as defined by their action on $\mathbb{C}^4$. We use the following decomposition:

$$\mathbb{C}^4 = \mathbb{C}_{00} \oplus \mathbb{C}_{01} \oplus \mathbb{C}_{11} \oplus \mathbb{C}_{10}. \tag{6}$$

We can then assign the following degrees:

$$\deg(\Sigma_0) = (0, 0), \quad \deg(\Sigma_1) = (0, 1), \tag{7}$$

$$\deg(\Sigma_2) = (0, 1), \quad \deg(\Sigma_3) = (0, 0), \tag{8}$$

$$\deg(\alpha_0) = (1, 1), \quad \deg(\alpha_1) = (1, 0), \tag{9}$$

$$\deg(\alpha_2) = (1, 0), \quad \deg(\alpha_3) = (1, 1). \tag{10}$$

This assignment of the $\mathbb{Z}_2^n$-degree will be essential in how we define the charges in our model.

Note: From now on, all commutators will be $\mathbb{Z}_2^2$-graded commutators unless otherwise stated [see (2)].

Proposition 7. *With the above $\mathbb{Z}_2^2$-grading [see (7)–(10)], the vector space (over $\mathbb{C}$) spanned by the Σ and α matrices forms a $\mathbb{Z}_2^2$-Lie algebra with respect to the commutator* (2).

Proof. First, we observe that the Σ and α matrices are linearly independent. Second, it is clear that we have closure under the ($\mathbb{Z}_2^2$-graded) commutator. This follows that all the properties of the Σ and α matrices are inherited from the Pauli matrices and that the Pauli matrices are closed under both (non-graded) (anti-)commutators. Third, the ($\mathbb{Z}_2^2$-graded) Jacobi identity is obviously satisfied, as we are dealing with the commutator. $\square$

Explicitly, the non-vanishing brackets are

$$[\Sigma_1, \Sigma_1] = 2\,\Sigma_0, \quad [\Sigma_1, \Sigma_3] = -2i\,\Sigma_2,$$
$$[\Sigma_2, \Sigma_2] = 2\,\Sigma_0, \quad [\Sigma_2, \Sigma_3] = 2i\,\Sigma_1,$$
$$[\alpha_0, \alpha_1] = 2\,\Sigma_1, [\alpha_0, \alpha_2] = 2\,\Sigma_2,$$
$$[\alpha_1, \alpha_1] = 2\,\Sigma_0, \quad [\alpha_2, \alpha_2] = 2\,\Sigma_0,$$
$$[\Sigma_1, \alpha_0] = 2\,\alpha_1, \quad [\Sigma_1, \alpha_2] = 2i\,\alpha_3,$$
$$[\Sigma_2, \alpha_0] = 2\,\alpha_2, \quad [\Sigma_2, \alpha_1] = -2i\,\alpha_3.$$

III. A DOUBLE-GRADED SUPERSYMMETRIC QUANTUM MECHANICAL SYSTEM

A. The specific model

In order to build a specific double-graded supersymmetric quantum mechanical system (see Definition 2), we propose the following operators acting on $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^4$:

$$Q_{01} = \frac{1}{2}\left(\frac{p}{\sqrt{2m}} \otimes \Sigma_1 + W \otimes \Sigma_2 \right), \tag{11}$$

$$Q_{10} = \frac{1}{2}\left(\frac{p}{\sqrt{2m}} \otimes \alpha_2 - W \otimes \alpha_1 \right), \tag{12}$$

$$H_{00} = \left(\frac{p^2}{2m} + W^2 \right) \otimes \Sigma_0 + \frac{\hbar}{\sqrt{2m}} W' \otimes \Sigma_3, \tag{13}$$

$$Z_{11} = -i \left(\left(\frac{p^2}{2m} + W^2 \right) \otimes \alpha_3 + \frac{\hbar}{\sqrt{2m}} W' \otimes \alpha_0 \right). \tag{14}$$

Here, $W := W(x) \in C^\infty(\mathbb{R})$ and $W' := \frac{dW}{dx}$. As standard $p = -i \frac{d}{dx}$, i.e., we are using the Schrödinger representation. We will further impose the requirement that $|W| \to \infty$ as $x \to \pm\infty$. In this way, we have only bound states, a discrete spectrum, and critically, all the wave functions belong to the Hilbert space $\mathcal{H}$, i.e., we do not have plane wave solutions. Furthermore, for simplicity, we will always work with the full real line and not the half-line or some interval. This will avoid us having to discuss boundary conditions for different subspaces of the Hilbert space.

Remark 8. One can relax the smoothness condition on W for just C^1; however, for convenience, we will insist on smoothness.

Theorem 9. *The operators defined above in* (11)–(14). *Satisfy the* $\mathbb{Z}_2^2$-*supertranslation algebra* (1). *In other words, we have a* $\mathbb{Z}_2^2$-*SUSY QM system (see Definition 2) defined by the Hilbert space* $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^4$ *and the operators* (11)–(14).

Proof. We prove the theorem via direct computations.

- First, we consider the self-commutator of Q_{01},

$$[Q_{01}, Q_{01}] = \frac{1}{2} \left(\frac{p^2}{2m} \otimes \Sigma_1^2 + W^2 \otimes \Sigma_2^2 \right)$$
$$+ \frac{1}{2} \left(\frac{p}{\sqrt{2m}} W \otimes \Sigma_1 \Sigma_2 + W \frac{p}{\sqrt{2m}} \otimes \Sigma_2 \Sigma_1 \right).$$

Now, we use $\Sigma_1^2 = \Sigma_2^2 = \Sigma_0$ and $[\Sigma_1, \Sigma_2] = 0$ (being careful with the $\mathbb{Z}_2^2$-degree) to obtain

$$[Q_{01}, Q_{01}] = \frac{1}{2} \left(\frac{p^2}{2m} + W^2 \right) \otimes \Sigma_0$$
$$+ \frac{1}{2} \left(\frac{-i\hbar}{\sqrt{2m}} W' \otimes \Sigma_1 \Sigma_2 \right).$$

Next, we use $-i\Sigma_1\Sigma_2 = \Sigma_3$, and we arrive at

$$[Q_{01}, Q_{01}] = \frac{1}{2} \left(\frac{p^2}{2m} + W^2 \right) \otimes \Sigma_0$$
$$+ \frac{1}{2} \left(\frac{\hbar}{\sqrt{2m}} W' \otimes \Sigma_3 \right),$$

as required.

- Next, we consider the self-commutator of Q_{10},

$$[Q_{10}, Q_{10}] = \frac{1}{2} \left(\frac{p^2}{2m} \otimes \alpha_2^2 + W^2 \otimes \alpha_1^2 \right)$$
$$- \frac{1}{4} \left(\frac{p}{\sqrt{2m}} W \otimes \alpha_2 \alpha_1 + W \frac{p}{\sqrt{2m}} \otimes \alpha_1 \alpha_2 \right).$$

Now, we use $\alpha_1^2 = \alpha_2^2 = \Sigma_0$ and $[\alpha_1, \alpha_2] = 0$ to obtain

$$[Q_{10}, Q_{10}] = \frac{1}{2} \left(\frac{p^2}{2m} + W^2 \right) \otimes \Sigma_0$$
$$+ \frac{1}{2} \left(\frac{i\hbar}{\sqrt{2m}} W' \otimes \alpha_2 \alpha_1 \right).$$

After using $i\alpha_2\alpha_1 = \Sigma_3$, we arrive at the required expression.

- Next, we need to consider the "mixed" commutator of the $\mathbb{Z}_2^2$-supercharges,

$$
\begin{aligned}
\left[Q_{10}, Q_{01}\right] = \frac{1}{4}\Bigg(& \frac{p^2}{2m} \otimes [\alpha_2, \Sigma_1] - W^2 \otimes [\alpha_1, \Sigma_2] \\
& + \frac{p}{\sqrt{2m}} W \otimes (\alpha_2 \Sigma_2 + \Sigma_1 \alpha_1) \\
& - W \frac{p}{\sqrt{2m}} \otimes (\alpha_1 \Sigma_1 + \Sigma_2 \alpha_2)\Bigg).
\end{aligned}
\tag{15}
$$

Now, we use

$$
\begin{aligned}
&[\alpha_2, \Sigma_1] = -2i\alpha_3, [\alpha_1, \Sigma_2] = 2i\alpha_3, (\alpha_2 \Sigma_2 + \Sigma_1 \alpha_1) \\
&= (\alpha_1 \Sigma_1 + \Sigma_2 \alpha_2) = 2\alpha_0
\end{aligned}
$$

and obtain

$$
\begin{aligned}
\left[Q_{10}, Q_{01}\right] = -\frac{i}{2}\Bigg(& \left(\frac{p^2}{2m} + W^2\right) \otimes \alpha_3 \\
& + \frac{\hbar}{\sqrt{2m}} W' \otimes \alpha_0\Bigg),
\end{aligned}
$$

as required.

- It is clear that $[Q_{01}, H] = [Q_{10}, H] = 0$ follows from the Jacobi identity for the commutators. Thus, the only thing that now requires checking is if that Z_{11} is indeed central. From the $\mathbb{Z}_2^2$-supersymmetry algebra and the Jacobi identity, we need only to show that $[Q_{01}, Z_{11}] = 0$ and $[Q_{10}, Z_{11}] = 0$. For notational ease, we define $H_0 := \frac{p^2}{2m} + W^2$.

$$
\begin{aligned}
\left[Q_{01}, Z_{11}\right] = -\frac{i}{2}\Bigg(& \frac{p}{\sqrt{2m}} H_0 \otimes \Sigma_1 \alpha_3 + \frac{\hbar}{2m} p W' \otimes \Sigma_1 \alpha_0 + W H_0 \otimes \Sigma_2 \alpha_3 \\
& + \frac{\hbar}{\sqrt{2m}} W W' \otimes \Sigma_2 \alpha_0 + H_0 \frac{p}{\sqrt{2m}} \otimes \alpha_3 \Sigma_2 + H_0 W \otimes \alpha_3 \Sigma_2 \\
& + \frac{\hbar}{2m} W' p \otimes \alpha_0 \Sigma_1 + \frac{\hbar}{\sqrt{2m}} W' W \otimes \alpha_0 \Sigma_2\Bigg) \\
= -\frac{i}{2}\Bigg(& \frac{1}{\sqrt{2m}}[p, W^2] \otimes \Sigma_1 \alpha_3 + \frac{2\hbar}{\sqrt{2m}} W W' \otimes \Sigma_2 \alpha_0 \\
& + \frac{1}{2m}[W, p^2] \otimes \Sigma_2 \alpha_3 + \frac{\hbar}{2m}(pW' + W'p) \otimes \Sigma_1 \alpha_0\Bigg) \\
= -\frac{i}{2}\Bigg(& \left(\frac{-i}{\sqrt{2m}}[p, W^2] + \frac{2\hbar}{\sqrt{2m}} W W'\right) \otimes \alpha_2 \\
& + \left(\frac{i}{2m}[W, p^2] + \frac{\hbar}{2m}(pW' + W'p)\right) \otimes \alpha_1\Bigg).
\end{aligned}
$$

Evaluation of the commutators shows that the above expression vanishes.
A similar computation gives

$$
\begin{aligned}
\left[Q_{10}, Z_{11}\right] = -\frac{i}{2}\Bigg(& \frac{i}{\sqrt{2m}}[p, W^2] - \frac{2\hbar}{\sqrt{2m}} W W'\Bigg) \otimes \Sigma_1 \\
& -\frac{i}{2}\Bigg(\frac{\hbar}{2m}(pW' + W'p) + \frac{i}{2m}[W, p^2]\Bigg) \otimes \Sigma_2.
\end{aligned}
\tag{16}
$$

Then, once again by evaluating the commutators, this expression vanishes. $\qquad\square$

We will make the canonical identification between a matrix O and $(1 \otimes O)$ as needed.

Proposition 10. The Hamiltonian and central charge are related by

$$
H_{00}\, \alpha_3 = -i\, Z_{11}.
$$

Proof. This follows as a simple consequence of the fact that $\Sigma_0 \alpha_3 = \alpha_3$ and $\Sigma_3 \alpha_3 = \alpha_0$. $\qquad\square$

B. Ladder-like operators

We define the two following operators:

$$A := \frac{ip}{\sqrt{2m}} + W, \ A^\dagger := -\frac{ip}{\sqrt{2m}} + W,$$

which are identical to those found in Witten's model. We then write the following:

$$Q_{01} = \frac{i}{2}\left(\begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -A \\ A^\dagger & 0 \end{pmatrix} \right) = \frac{i}{2} \begin{pmatrix} 0 & -A & 0 & 0 \\ A^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & -A \\ 0 & 0 & A^\dagger & 0 \end{pmatrix},$$

$$Q_{10} = -\frac{1}{2}\left(\begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \ominus \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \right) = -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & A \\ 0 & 0 & A^\dagger & 0 \\ 0 & A & 0 & 0 \\ A^\dagger & 0 & 0 & 0 \end{pmatrix}.$$

Let us define $H_+ := \frac{p^2}{2m} + W^2 + \frac{\hbar}{\sqrt{2m}} W'$ and $H_- := \frac{p^2}{2m} + W^2 - \frac{\hbar}{\sqrt{2m}} W'$. A quick computation shows that $H_+ = AA^\dagger$ and $H_- = A^\dagger A$. Using this, we can obtain

$$H_{00} = \begin{pmatrix} H_+ & 0 & 0 & 0 \\ 0 & H_- & 0 & 0 \\ 0 & 0 & H_+ & 0 \\ 0 & 0 & 0 & H_- \end{pmatrix}, \tag{17}$$

$$Z_{11} = -i \begin{pmatrix} 0 & 0 & H_+ & 0 \\ 0 & 0 & 0 & -H_- \\ H_+ & 0 & 0 & 0 \\ 0 & -H_- & 0 & 0 \end{pmatrix}. \tag{18}$$

C. Parity operators

The Hilbert space we are considering has a natural decomposition,

$$\mathcal{H} = \mathcal{H}_{00} \oplus \mathcal{H}_{01} \oplus \mathcal{H}_{11} \oplus \mathcal{H}_{10}.$$

As we have a bi-grading, naturally we have a pair of parity operators (by definition these must be degree $(0,0)$ operators). Explicitly,

$$\mathbb{K}_1 = \sigma_0 \oplus (-\sigma_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{19}$$

$$\mathbb{K}_2 = \sigma_3 \oplus \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

Direct computation establishes the following:

Proposition 11. The pair of parity operators $\mathbb{K}_1$ and $\mathbb{K}_2$ satisfy the following relations:

$$[\mathbb{K}_1, \mathbb{K}_2] = 0, \tag{21}$$

$$(\mathbb{K}_1)^2 = (\mathbb{K}_2)^2 = \Sigma_0. \tag{22}$$

Moreover,

$$[\mathbb{K}_1, H_{00}] = [\mathbb{K}_2, H_{00}] = 0. \tag{23}$$

Corollary 12. Simultaneous eigenstates of the Hamiltonian H_{00} and the parity operators $\mathbb{K}_1$ and $\mathbb{K}_2$ exist.

The above result will be essential in describing explicitly how the $\mathbb{Z}_2^2$-supercharges act on energy eigenstates.

From direct calculation, we observe (rather naturally) that

$$\mathbb{K}_1 Q_{01} = + Q_{01} \mathbb{K}_1, \qquad\qquad \mathbb{K}_2 Q_{01} = - Q_{01} \mathbb{K}_2, \qquad (24)$$

$$\mathbb{K}_1 Q_{10} = - Q_{01} \mathbb{K}_1, \qquad\qquad \mathbb{K}_2 Q_{10} = + Q_{10} \mathbb{K}_2, \qquad (25)$$

$$\mathbb{K}_1 Z_{11} = - Z_{11} \mathbb{K}_1, \qquad\qquad \mathbb{K}_2 Z_{11} = - Z_{11} \mathbb{K}_2. \qquad (26)$$

Remark 13. In the definition of a $\mathbb{Z}_2^2$-SUSY QM system (see Definition 2), we do not postulate the existence of the parity operators that commute with the Hamiltonian. However, the existence of such operators is essential in describing $\mathbb{Z}_2^2$-supersymmetry as a mapping between states of different $\mathbb{Z}_2^2$-degrees.

D. Energy eigenstates and $\mathbb{Z}_2^2$-supersymmetry

It is clear that

$$Q_{01} \mathcal{H}_{ij} \subset \mathcal{H}_{i(j+1)}, \, Q_{10} \mathcal{H}_{ij} \subset \mathcal{H}_{(i+1)j}. \qquad (27)$$

Due to Corollary 12, we can present the action of the $\mathbb{Z}_2^2$-supercharges on energy eigenstates rather explicitly and construct the corresponding multiplets. In particular, we can label the simultaneous eigenstates of H_{00}, $\mathbb{K}_1$, and $\mathbb{K}_2$ as

$$|E, i, j\rangle \in \mathcal{H}_{ij}.$$

Proposition 14. For every state $|E, i, j\rangle$, with energy eigenvalue $E > 0$, there exist two other states $|E, i, j + 1\rangle$ and $|E, i + 1, j\rangle$, with the same energy eigenvalue E.

Proof. We define $|E, i, j + 1\rangle := \frac{2}{\sqrt{E}} Q_{01} |E, i, j\rangle$. Now, we must check that this really is an energy eigenstate with eigenvalue E. This follows as $[H, Q] = 0$. Explicitly,

$$H_{00} |E, i, j + 1\rangle = H_{01} \left(\frac{2}{\sqrt{E}} Q_{01} |E, i, j\rangle \right)$$

$$= \frac{2}{\sqrt{E}} Q_{01} (H_{00} |E, i, j\rangle)$$

$$= \frac{2}{\sqrt{E}} Q_{01} (E |E, i, j\rangle) = E |E, i, j + 1\rangle.$$

Similarly, we define $|E, i + 1, j\rangle := \frac{2}{\sqrt{E}} Q_{10} |E, i, j\rangle$, and an almost identical calculation to the above shows that this state is also an energy eigenstate of energy E. □

Remark 15. Proposition 14 does not depend on the specific model, but only the existence of the pair of parity operators that commute with the Hamiltonian.

Now, let us proceed to the specifics of this model and define the corresponding wave function in the position representation,

$$\Phi_{ij}^E(x) := \langle x | E, i, j \rangle.$$

In particular,

$$\Phi_{00}^E(x) = \begin{pmatrix} \psi_{00}^E(x) \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\psi_{00}^E(x) \in L^2(\mathbb{R})$. Clearly, it is an eigenvector of H_+, i.e., $H_+ \psi_{00}^E = E \psi_{00}^E$. Similar expressions hold for the other degree components. We again assume the energy to be strictly greater than zero. We define the $\mathbb{Z}_2^2$-multiplet generated by Φ_{00}^E as all the states that can be "reached" by application of Q_{01} and Q_{10}. We will normalize the operators for convenience and define

$$\overline{Q}_{01} := \frac{2}{\sqrt{E}} Q_{01}, \, \overline{Q}_{10} := \frac{2}{\sqrt{E}} Q_{10}.$$

Direct computation gives

$$
\overline{Q}_{01}\Phi_{00}^{E}(x) = \begin{pmatrix} 0 \\ \frac{i}{\sqrt{E}}A^{\dagger}\psi_{00}^{E}(x) \\ 0 \\ 0 \end{pmatrix},
\tag{28}
$$

$$
\overline{Q}_{10}\Phi_{00}^{E}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{\sqrt{E}}A^{\dagger}\psi_{00}^{E}(x) \end{pmatrix},
\tag{29}
$$

$$
\overline{Q}_{10}\overline{Q}_{01}\Phi_{00}^{E}(x) = \begin{pmatrix} 0 \\ 0 \\ -i\psi_{00}^{E}(x) \\ 0 \end{pmatrix},
\tag{30}
$$

$$
\overline{Q}_{01}\overline{Q}_{10}\Phi_{00}^{E}(x) = \begin{pmatrix} 0 \\ 0 \\ i\psi_{00}^{E}(x) \\ 0 \end{pmatrix}.
\tag{31}
$$

Note that the last two states are clearly not linearly independent. Thus, the $\mathbb{Z}_2^2$-multiplet generated by Φ_{00}^{E} is the sub-vector space of $\mathcal{H}$ spanned by the following elements:

$$
\begin{pmatrix} \psi_{00}^{E}(x) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ A^{\dagger}\psi_{00}^{E}(x) \\ 0 \\ 0 \end{pmatrix},
\tag{32}
$$

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \\ A^{\dagger}\psi_{00}^{E}(x) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \psi_{00}^{E}(x) \\ 0 \end{pmatrix}.
\tag{33}
$$

Remark 16. Similar conclusions can be drawn by looking at $\mathbb{Z}_2^2$-multiplets generated by another reference states.

Clearly, we have the following:

Proposition 17. Energy levels with E > 0 are fourfold degenerate.

E. Zero energy states

To examine the nature of the zero energy states, let us consider the following states (written in the position representation):

$$
\begin{pmatrix} \psi_{00}(x) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_{01}(x) \\ 0 \\ 0 \end{pmatrix},
\tag{34}
$$

$$
\begin{pmatrix} 0 \\ 0 \\ \psi_{11}(x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi_{10}(x) \end{pmatrix}.
\tag{35}
$$

Each component belongs to $L^2(\mathbb{R})$. Applying Q_{01} and separately Q_{10} to these states and insisting that the result vanishes give the conditions,

$$
A^{\dagger}\psi_{00} = 0, \qquad\qquad\qquad A^{\dagger}\psi_{11} = 0,
\tag{36}
$$

$$A\psi_{01} = 0, \qquad\qquad A\psi_{10} = 0. \qquad (37)$$

We know from the analysis of Witten's model[7] that all these conditions cannot be consistent for non-trivial wave functions—some of them or all of these wave functions must be zero. Thus, the zero energy states (in the position representation) are either of the form

$$X(x) = a\begin{pmatrix} \chi(x) \\ 0 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ \chi(x) \\ 0 \end{pmatrix} \in \mathcal{H}_{00} \oplus \mathcal{H}_{11} \qquad (38)$$

or

$$\Psi(x) = c\begin{pmatrix} 0 \\ \psi(x) \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi(x) \end{pmatrix} \in \mathcal{H}_{01} \oplus \mathcal{H}_{10}, \qquad (39)$$

where $A^\dagger\chi = 0$, $A\psi = 0$, and a, d, c and $d \in \mathbb{C}$.

We have thus proved the following:

Proposition 18. A zero energy state is either non-existent or twofold degenerate and belongs to either $\mathcal{H}_{00} \oplus \mathcal{H}_{11}$ or $\mathcal{H}_{01} \oplus \mathcal{H}_{10}$, i.e., to either the even or odd subspaces of $\mathcal{H}$.

F. The harmonic oscillator potential

Note that the spectrum of our $\mathbb{Z}_2^2$-SUSY QM model is identical to that of the corresponding Witten model. The key difference is in the degeneracy of the energy eigenstates: everything is doubled. In order to explicitly illustrate the structure of our model, we will examine the potential,

$$W(x) = -\sqrt{\frac{m}{2}}\omega x \qquad (40)$$

where $\omega > 0$. Clearly, we have

$$H_+ = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 - \frac{\hbar}{2}\omega, \qquad (41)$$

$$H_- = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + \frac{\hbar}{2}\omega. \qquad (42)$$

We have chosen the sign in the potential so that the zero energy states belong to $\mathcal{H}_{00} \oplus \mathcal{H}_{11}$. This is just for our convenience. We see that the Hamiltonians are just the standard Hamiltonians for the harmonic oscillator with constant shifts by one unit of the energy. Thus, the wave functions are just the usual wave functions for the harmonic oscillator. As standard, we define

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{mw}{2\hbar}x^2\right) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right),$$

where H_n is the nth Hermite polynomial. It is straightforward to see that the spectrum of H_{00} with the potential (40) is

$$0, \ \hbar\omega, \ 2\hbar\omega, \ 3\hbar\omega, \ \cdots.$$

This is, of course, identical to the Witten model. The zero energy ground states are (weighted) sums of the vectors

$$\begin{pmatrix} \psi_0(x) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \psi_0(x) \\ 0 \end{pmatrix}. \qquad (43)$$

The excited states ($n \geq 1$) are then (weighted) sums of the vectors

$$\begin{pmatrix} \psi_n(x) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_{n-1}(x) \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ \psi_n(x) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \psi_{n-1}(x) \end{pmatrix}. \tag{44}$$

We see the twofold degeneracy in a zero energy ground state and the fourfold degeneracy in the excited states.

IV. CONCLUSIONS

Thus, we have constructed a double-graded supersymmetric quantum mechanical model in order to illustrate that employing "higher gradings" in physics and, in particular, quantum mechanics is possible and can lead to interesting results. Specifically, we have a realization of the $\mathbb{Z}_2^2$-supertranslation algebra, as proposed in Ref.18 via a quantum mechanical model. This model is not equivalent to the models with para-Grassmann variables[16,17] and parasupersymmetries[13,38] or with various versions of the (extended) supersymmetric quantum mechanics.[12,27,54,55] Indeed, the direct physical interpretation of $\mathbb{Z}_2^2$-SUSY QM is currently lacking. We expect more involved models of $\mathbb{Z}_2^2$-supersymmetric quantum mechanics, i.e., "multiple-graded supersymmetric quantum mechanics" to be important and lead to more unexpected results.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

[1] Yu. A. Golfand and E. P. Likhtman, JETP Lett. **13**, 323 (1971); Pisma Z. Eksp. Teor. Fiz. **13**, 452 (1971).

[2] D. V. Volkov and V. P. Akulov, JETP Lett. **16**, 438 (1972).

[3] J. Wess and B. Zumino, Nucl. Phys. B **70**, 39 (1974).

[4] *Concise Encyclopedia of Supersymmetry and Noncommutative Structures in Mathematics and Physics*, edited by S. Duplij, W. Siegel, and J. Bagger (Kluwer Academic Publishers, Dordrecht, Boston, London, 2004), second printing [Springer Science and Business Media, Berlin, New York, Heidelberg, (2005)].

[5] *The Supersymmetric World: The Beginnings of the Theory*, edited by G. L. Kane and M. Shifman (World Scientific, Singapore, 2000).

[6] K. Efetov, *Supersymmetry in Disorder and Chaos* (Cambridge University Press, Cambridge, 1997), p. 441.

[7] E. Witten, Nucl. Phys. B **188**, 513 (1981).

[8] B. K. Bagchi, *Supersymmetry in Quantum and Classical Mechanics*, Monographs and Surveys in Pure and Applied Mathematics Vol. 116 (Chapman & Hall/CRC, Boca Raton, FL, 2001), p. xiv+222.

[9] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry in Quantum Mechanics* (World Scientific Publishing Co., Inc., River Edge, NJ, 2001), p. xii+210.

[10] C. V. Sukumar, AIP Conf. Proc. **744**, 166 (2004).

[11] J. Szczęsny, Fortsch. Phys. **49**, 723 (2001).

[12] S. Fubini and E. Rabinovici, Nucl. Phys. B **245**, 17 (1984).

[13] V. A. Rubakov and V. P. Spiridonov, Mod. Phys. Lett. A **03**, 1337 (1988).

[14] A. Khare, A. K. Mishra, and G. Rajasekaran, Int. J. Mod. Phys. **08**, 1245 (1993).

[15] S. Durand, Phys. Lett. B **312**, 115 (1993).

[16] V. D. Gershun and V. I. Tkach, Ukr. Fiz. Zh. **29**, 1620 (1984).

[17] V. D. Gershun and V. I. Tkach, Probl. Nucl. Phys. Cosmic Rays **23**, 42 (1985).

[18] A. Bruce, Symmetry **11**, 116 (2019).

[19] V. Molotkov, "Infinite-dimensional $\mathcal{F}_2^{-k}$ supermanifolds," ICTP preprint IC/84/183 (Trieste, 1984).

[20] V. Molotkov, J. Nonlinear Math. Phys. **17**, 375 (2010).

[21] T. Covolo, J. Grabowski, and N. Poncin, J. Math. Phys. **57**, 073503 (2016).

[22] T. Covolo, J. Grabowski, and N. Poncin, J. Geom. Phys. **110**, 393 (2016).

[23] V. N. Tolstoy, in *Lie Theory and Its Application in Physics*, edited by V. Dobrev (Springer, Tokyo, 2014), pp. 357–367.

[24] F. A. Berezin, *Introduction to Superanalysis* (D. Reidel, Dordrecht, 1987), p. 421.

[25] D. A. Leites, Russ. Math. Surv. **35**, 1 (1980).

[26] D. A. Leites, *Supermanifold Theory* (Math. Methods Sci. Invest., Petrozavodsk, 1983).

[27] V. P. Akulov and A. I. Pashnev, Theor. Math. Phys. **56**, 862 (1983).

[28] L. E. Gendenshtein and I. V. Krive, Sov. Phys. Usp. **28**, 645 (1985).

[29] V. D. Gershun and V. I. Tkach, JETP Lett. **29**, 288 (1979).

[30] A. I. Pashnev, Theor. Math. Phys. **69**, 1172 (1986).

[31] V. Akulov and M. Kudinov, Phys. Lett. B **460**, 365 (1999).

[32] A. Pashnev and F. Toppan, J. Math. Phys. **42**, 5257 (2001).

[33] N. Aizawa, Z. Kuznetsova, H. Tanaka, and F. Toppan, Prog. Theor. Exp. Phys. **2016**, 123A01.

[34] H. S. Green, Phys. Rev. **90**, 270 (1953).

[35] D. V. Volkov, Sov. Phys. JETP **9**, 1107 (1959).

[36] V. N. Tolstoy, Phys. Part. Nucl. Lett. **11**, 933 (2014).

[37] J. Beckers and N. Debergh, J. Phys. A: Math. Gen. **23**, L751S (1990).

[38] J. Beckers and N. Debergh, J. Phys. A: Math. Gen. **24**, L597 (1991).

[39] A. Khare, J. Phys. A: Math. Gen. **25**, L749 (1992).

[40] D. Leites and V. Serganova, Phys. Lett. B **252**, 91 (1990).

[41] D. Leites and V. Serganova, in *Noncommutative Structures in Mathematics and Physics*, edited by S. Duplij and J. Wess (Kluwer, Dordrecht, 2001), pp. 13–30.

[42] A. A. Baranov, Sib. Math. J. **36**, 859 (1995).

[43] I. B. Volichenko, "Nonhomogeneous subalgebras of commutative superalgebras," [Preprint, No. 26] [in Russian], Akad. Nauk BSSR, Minsk (1985).

[44] M. Scheunert, J. Math. Phys. **20**, 712 (1979).

[45] J. Bernstein, D. Leites, V. Molotkov, and V. Shander, *Seminars of Supersymmetries: Volume 1. Algebra and Calculus*, edited by D. A. Leites (MCCME, Moscow, 2011) (in Russian), the English version is available for perusal.

[46] T. Covolo, V. Ovsienko, and N. Poncin, J. Geom. Phys. **62**, 2294 (2012).

[47] S. Morier-Genoud and V. Ovsienko, J. Algebra **323**, 1649 (2010).

[48] V. Rittenberg and D. Wyler, Nucl. Phys. B **139**, 189 (1978).

[49] J.-L. Loday, Enseign. Math. **39**(2), 269 (1993).

[50] M. Faux and D. Spector, Phys. Rev. D **70**, 085014 (2004).

[51] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto, K. Tatsumi, and I. Ueba, Phys. Rev. D **99**, 065002 (2019).

[52] J. Niederle and A. G. Nikitin, in *Symmetry in Nonlinear Mathematical Physics, Part 1, 2* (Kyiv, 2001), Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 43, Part 1, Vol. 2 (Natsīonal. Akad. Nauk Ukraïni, Inst. Mat., Kiev, 2002), pp. 497–507.

[53] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists* (Elsevier Academic Press, Burlington, MA, 2005).

[54] V. P. Akulov, O. Cebecioglu, and A. Pashnev, Phys. Lett. **562**, 132 (2003).

[55] V. Akulov and S. Duplij, in *Supersymmetries and Quantum Symmetries*, edited by J. Wess and E. A. Ivanov (Springer, Berlin, Heidelberg, 1999), pp. 235–242.